Unofficial Practice Test 2

for Math 1X03

Fall 2022

1 A Brief Preface

The idea of this collection of problems is to give a structured set of practice problems that survey the midterm material. We have attempted to list problems that are a similar level of difficulty to tests for previous instances of this course and the practice problems from the textbook.

The structure of this document is as follows: In Section 2, we provide a breakdown of topics that *may* appear on the midterm. In Section 3, we present a collection of questions which sample the midterm content at a level of difficulty that is *hopefully* reasonably similar to what you will see on the test. Section 4 gives true/false questions and Section 5 gives prompts for topics to review (non-exhaustive). In Section 6, we provide a list of challenging questions, which are almost certainly more difficult than what you were encounter. They are provided for if the reader would like to challenge themselves as well as to provide examples of further applications of the course material. Section 7 gives hints for selected exercises.

Since we as students do not have an infinite amount of time with which to study, we would suggest focusing on Section 3. Upon completing these questions, test your understanding in Section 4. After that, have a read through of Section 6, and try one or two of these problems if you like, but your time is likely better spent on the assignments and the practice problems on Avenue.

2 Midterm content

The midterm will cover all topics seen in lecture up to (and including) the lecture on Tuesday, November 1. This covers content form Chapters 1 through 4 (inclusive). *We would expect* that the focus of the content to be that which has been covered since Midterm 1. The following table is an *approximate* list of the topics which were not on the first midterm, but are on the second midterm.

Text Section	**Topics**
3.5	Higher-order derivatives
3.6	Derivatives of trig functions
3.7	The chain rule
3.8	Implicit differentiation and the derivative of inverse functions
3.9	Derivatives of logarithmic and exponential functions
4.2	Extreme values
4.3	The Mean Value Theorem and monotonicity
4.4	The second derivative and concavity
4.5	L'Hôpital's Rule

3 Sample questions

1. Which of the following satisfy $f^{(k)}(x) = 0$ for all $k \geq 6$?

(a) $f(x) = 7x^4 + 4 + x^{-1}$ (b) $f(x) = \sqrt{x}$ (c) $f(x) = x^{9/5}$

(d) $f(x) = x^3 - 2$ (e) $f(x) = 1 - x^6$ (f) $f(x) = 2x^2 + 3x^5$

2. Calculate the first five derivatives of $f(x) = \cos x$. Then determine $f^{(2022)}(x)$, $f^{(2023)}(x)$, $f^{(2024)}(x)$, and $f^{(2025)}(x)$.

3. Assume that g is a differentiable function. Compute $f'(x)$ for $f(x) = \ln(\sin(e^{g(x)}) + 11)$.

4. Find the equation of the tangent line to $y^2(y^2 - 4) = x^2(x^2 - 5)$ at $(0, -2)$. *(This curve is known as the "devil's curve". Plug it into an online grapher!)*

5. Find the absolute and local extrema of $f(x) = x + \frac{1}{x}$ on the interval $[0.2, 4]$.

6. (a) Show that $f(x) = \ln x$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 5]$.

(b) Suppose that g is differentiable, $g(0) = 1$, and $g'(x) \geq 3$ for all $x \in (0, 10)$. According to the Mean Value Theorem, what is the smallest possible value of $g(10)$?

7. Let $f(x) = 5x^3 - 3x^5$.

(a) Find the roots of f.

(b) Find the local and global extrema of f, if any exist.

(c) Find the intervals of increase and decrease of f.

(d) Find the inflection points and the intervals on which f is concave up and concave down.

(e) Sketch the graph of f.

8. Evaluate $\lim_{x \to \infty} (e^x - x)$.

4 True or false questions

Determine whether each is true or false. If true, provide a brief justification. If false, provide a counterexample.

TF1. If $y = e^2$ then $y' = 2e$.

TF2. If f and g are differentiable functions and $f(x) \neq 0$, then $\dfrac{d}{dx}\dfrac{g(x)}{f(x)} = \dfrac{-f'(x)g(x) + g'(x)f(x)}{(f(x))^2}$.

TF3. $\dfrac{d}{dx}|x^2 + x| = |2x + 1|$.

TF4. If $f''(c) = 0$ then $(c, f(c))$ is an inflection point of f.

TF5. If f and g are increasing functions, then fg is increasing.

TF6. If $f'(x)$ exists an is nonzero for all x, then $f(0) \neq f(1)$.

TF7. If f is periodic, then f' is periodic.

TF8. If f is continuous on (a, b), then it attains a maximum and minimum value on (a, b).

5 Review

Attempt these problems from memory before checking with the textbook or any notes. Note that this is **not** an exhaustive list of definitions and theorems that may appear on the test.

R1. State the following derivative rules.

(a) power rule (b) sum rule (c) quotient rule
(d) constant rule (e) product rule (f) chain rule

R2. What are the derivatives of the following trig functions?

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\sec x$ (e) $\cot x$ (f) $\csc x$

R3. What are the derivatives of the following functions?

(a) e^x (b) b^x (c) $\ln x$ (d) $\log_b x$

R4. When is a point a critical point? Is a critical point also a local extrema? Is a local extrema also a critical point?

R5. When is a point an inflection point?

R6. What does Rolle's Theorem say? (Include both the necessary hypotheses and its conclusion.)

R7. What does the Intermediate Value Theorem say?

R8. What does the Mean Value Theorem say?

R9. If $f'(x) = 0$ for all x in an interval $x \in (a, b)$, what can we say about f on this interval?

R10. If $f'(x) > 0$, what can we say about $f(x)$? What if $f'(x) < 0$? What if $f'(x)$ does not exist?

R11. If $f''(x) > 0$, what can we say about $f(x)$? What if $f''(x) < 0$?

R12. Suppose that c is a critical point of $f(x)$ and $f''(c)$ exists. What can we say about the critical point if $f''(c) > 0$? What about when $f''(c) < 0$? Or $f''(c) = 0$?

6 Challenge questions

Note: Question C3 is the easiest in this section.

C1. Assume that temperature is a continuous function and assign to each point on a circle a temperature value. Show that there exist diametrically opposed points whose temperatures are equal.

[Note: "Diametrically opposed" means that the points are diagonally opposite from one another. For instance, on the unit circle, the points $(1, 0)$ *and* $(-1, 0)$ *are diametrically opposed.]*

C2. Use the Mean Value Theorem to show that $|\sin a - \sin b| \leq |a - b|$ for all a and b.

C3. Show that a cubic polynomial always has exactly one inflection point.

C4. Show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$. **Note:** this question requires the use of mathematical induction.

7 Hints for selected problems

We strongly recommend attempting problems before referring to this section.

7.1 Sample questions

1. There are exactly two correct answers.
5. There is a global maximum, a global minimum, and additionally, another local maximum.
7. Check your work using an online grapher.
8. Use l'Hôpital's rule.

7.2 True or false questions

TF1. Graph it.

TF3. Graph it and remember that a decreasing graph corresponds to a negative derivative. What is the derivative of $|x|$?

TF4. What about x^3 at $c = 0$?

TF5. Compute the derivative. Can it be negative?

7.3 Challenge questions

C1. Assume that the circle is centred at the origin and identify each point on the circle by an angle $0 \leq \theta < 2\pi$. Notice that diametrically opposed points differ by an angle of π. If $T(\theta)$ is our temperature function, then the function we want to consider is $t(\theta) = T(\theta) - T(\theta + \pi)$, which is difference in temperature of diametrically opposed points. Use the fact that T is continuous to conclude that t is continuous, and apply the Intermediate Value Theorem.

C2. The Mean Value Theorem guarantees that there exists some c such that $\sin(a) - \sin(b) = \cos(c)(a - b)$, or equivalently, $\cos(c) = \frac{\sin(a)-\sin(b)}{a-b}$. Why? What can you conclude?