Math 2R03 Exam Review
19 April 2023

Contents

1 Knowledge Questions **1**

2 Proof Questions **4**

1 Knowledge Questions

The questions in this section are designed to test recall of basic definitions and properties. You should be able to answer them with only a moment's thought. An answer key is at the end of the section.

Question 1 (Subspaces)**.** Let U and W be subspaces of a vector space V. Which of the following is always true?

(a) Both $U \cap W$ and $U \cup W$ are subspaces of V
(b) Only $U \cap W$ is a subspace of V
(c) Only $U \cup W$ is a subspace of V
(d) Neither $U \cap W$ nor $U \cup W$ is a subspace of V.

Question 2 (Isomorphisms)**.** We will write $V \cong W$ if V and W are isomorphic. Which of the following is **true**? Consider all vector spaces as **real** vector spaces.

(a) $\mathbb{R}^n \cong \mathbb{C}^n$ and $\mathbb{R}^n \cong \mathcal{P}_n(\mathbb{R})$
(b) $\mathbb{R}^n \cong \mathbb{C}^n$ and $\mathbb{R}^n \cong \mathcal{P}_{n+1}(\mathbb{R})$
(c) $\mathbb{R}^n \cong \mathbb{C}^{2n}$ and $\mathbb{R}^n \cong \mathcal{P}_{n-1}(\mathbb{R})$
(d) $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and $\mathbb{R}^{n+1} \cong \mathcal{P}_n(\mathbb{R})$

Question 3 (Fundamental Theorem of Algebra)**.** Let $p \in \mathcal{P}(\mathbb{R})$ be a polynomial with **real** coefficients. Which of the following is **not** a possibility for the number of complex roots of p?

(a) 0 (b) 1 (c) 2 (d) 4

Question 4 (Self-adjoint operators)**.** Suppose $T \in \mathcal{L}(\mathbb{C}^2)$ has matrix (with respect to the standard basis) given by $\begin{bmatrix} 2 & k \\ 3 & 7 \end{bmatrix}$. Then T is self-adjoint for which value(s) of k?

Question 5 (Matrices)**.** Suppose a linear transformation $T \in \mathcal{L}(V, W)$ has a matrix (with respect to some basis) that has a column of zeros. Which of the following is **always true**?

(a) T is neither injective nor surjective

(b) T is not injective

(c) T is not surjective

(d) T is both injective and surjective

Question 6 (Duals). Assume V and W are finite-dimensional vector spaces. Let $T \in \mathcal{L}(V, W)$ and take U to be a subspace of V. Which of the following is **false**?

(a) If T is surjective, then $T' = 0$

(b) The dual space V' is a vector space

(c) $\dim V = \dim V'$

(d) $\dim U + \dim U^\circ = \dim V$

Question 7 (Quotients). Let V be a finite-dimensional vector space and U a subspace of U. What is the dimension of V/U?

Question 8 (Inner products). Consider the map $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ given by

$$(p, q) = \int_0^\infty p(x)q(x)e^{-x}\, dx$$

for any $p, q \in \mathcal{P}(\mathbb{R})$. Is this map an inner product? If not, which property/properties from the definition is/are not satisfied?

Question 9 (Normal operators). Let V be a finite-dimensional **real** inner product space. Define

$$A := \{T \in \mathcal{L}(V) \mid T \text{ is self-adjoint}\},$$
$$N := \{T \in \mathcal{L}(V) \mid T \text{ is normal}\}.$$

Is $A \subset N$ or $N \subset A$ (or neither)? Are either a subspace of $\mathcal{L}(V)$? What if V is a complex vector space?

Question 10 (Diagonalization). Let V be a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Which of the following is **not** equivalent to the others?

(a) T is diagonalizable

(b) V has a basis of eigenvectors of T

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

(d) T has $\dim V$ distinct eigenvalues

Question 11 (Eigenvalues). Consider the "shift" operators $R, L \in \mathcal{L}(\mathbb{C}^\infty)$ given by

$$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots),$$
$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Find an eigenvalue of each, if one exists.

Question 12 (Dimension). Compute the dimension of the following **real** vector spaces.

(i) $\mathrm{Mat}_{n\times m}(\mathbb{C})$ (ii) $\mathcal{P}_{n+1}(\mathbb{C})$ (iii) $\dim \mathbb{C}^n$

Question 13 (Orthogonal complements)**.** Let V be a finite-dimensional inner product space V with subspace U. Which of the following is **false**?

(a) $\mathrm{null}\, P_U = U^\perp$

(b) $\dim U^\perp = \dim V - \dim U$

(c) $\dim P_U^2 = P_U$

(d) $\|P_U v\| = \|v\|$ for all $v \in V$

Question 14 (Linear maps)**.** Which of the following is **not** a linear map on $\mathcal{P}(\mathbb{R})$?

1. $Tp = p'$
2. $Tp = 3p$
3. $Tp = xp$
4. $Tp = p^2$

Question 15 (Spectral Theorem)**.** Let V be a **complex** inner product space and $T \in \mathcal{L}(V)$. Which of the following is **not** an equivalence guaranteed by the Complex Spectral Theorem?

(a) T is self-adjoint

(b) T is normal

(c) V has an orthonormal basis consisting of eigenvectors of T

(d) T has a diagonal matrix with respect to some orthonormal basis of V.

---

answers follow

Answers

1. (b)
2. (d)
3. (b)
4. $k = 3$
5. (b)
6. (a)
7. $(\dim V) - (\dim U)$
8. It is an inner product
9. $A \subset N$ and A is a subspace over $\mathbb{R}$ but not $\mathbb{C}$
10. (d)
11. R has no eigenvalues; $\lambda = 1$ for L
12. (i) $2nm$ (ii) $2(n+2)$ (iii) $2n$
13. (d)
14. (d)
15. (a)

2 Proof Questions

The questions in this section are exercises from the suggested problems/assignments/tutorials. Sketches of solutions/proofs are provided. These sketches ask "why?" whenever additional justification is needed. **When reading these sketches, you should try and fill in these details.** You should try the problems first before looking at the sketches below.

The exercises covered are:

- 3.E.13
- 3.F.9
- 5.B.3
- 5.C.1
- 6.A.7
- 6.B.1
- 6.C.11
- 7.A.2

Exercise (Axler 3.E.13)**.** Suppose U is a subspace of V, $v_1 + U, \dots, v_m + U$ is a basis of V/U, and $u_1, \dots, u_n$ is a basis of U. Prove that $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V.

Solution. We are required to show that the list both spans V and is linearly independent.

To see that it spans, let $w \in V$ and consider $w + U$. Then

$$w + U = (c_1v_1 + \cdots + c_mv_m) + U$$

for some $c_1, \dots, c_m$. (Why?) Then $w - (c_1v_1 + \cdots + c_mv_m) \in U$ (why?) and so we have

$$w - (c_1v_1 + \cdots + c_mv_m) = k_1u_1 + \cdots + k_nu_n$$

for some $u_1, \dots, u_n$. Hence

$$w = c_1v_1 + \cdots c_mv_m + k_1u_1 + \cdots + k_nu_n$$

and so $\operatorname{span}(v_1, \dots, v_m, u_1, \dots, u_n) = V$.

For linear independence, suppose that

$$c_1v_1 + \cdots + c_mv_m + k_1u_1 + \cdots + k_nu_n = 0.$$

This implies that

$$c_1v_1 + \cdots + c_mv_m = -(k_1u_1 + \cdots + k_nu_n),$$

so in particular, we notice that

$$c_1v_1 + \cdots + c_mv_m \in U.$$

This implies that

$$c_1(v_1 + U) + \cdots + c_m(v_m + U) = 0 + U.$$

(Why?) So we conclude that $c_1 = \cdots = c_m = 0$, which in turn implies that $k_1 = \cdots = k_n = 0$. ◊

The following exercise gives a way to write a given linear functional as a linear combination of the dual basis.

Exercise (Axler 3.F.9)**.** Suppose that $v_1, \dots, v_n$ is a basis of V and $\varphi_1, \dots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

Solution. There exist scalars $c_1, \dots, c_n$ such that $\psi = c_1\varphi_1 + \cdots + c_n\varphi_n$. (Why?) So it remains to show that $c_i = \psi(v_i)$ for all i. Fix some i and compute:

$$\psi(v_i) = c_i \cdot \varphi_i(v_i) = c_i.$$

(Why?) So $\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$. ♢

Exercise (5.B.3)**.** Suppose $T \in \mathcal{L}(V)$ and $T^2 = I$ and -1 is not an eigenvalue of T. Prove that $T = I$.

Solution. Since $T^2 = I$ we have $T^2 - I = 0$ or equivalently, $(T - I)(T + I) = 0$. We conclude (why?) that $T - I = 0$ (i.e., $(T - I)v = 0$ for all $v \in V$) so $T = I$. ♢

Exercise (5.C.1)**.** Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$.

Solution. The result is clear if T is injective. (Why?)

So we treat the case that T is not injective. Since T is diagonalizable, we have that

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m),$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of T. Note that 0 is an eigenvalue of T and in particular, we have $\operatorname{null} T = E(0, T)$. (Why?) Write $\lambda_1 = 0$ so label the remaining, nonzero eigenvalues (if they exist) are $\lambda_2, \dots, \lambda_m$. It suffices to show that $\operatorname{range} T = E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(⊆) If $w \in \operatorname{range} T$ then there is some $v \in V$ such that $Tv = w$. Since the eigenspaces form a direct sum of V, we have that $v = c_1v_1 + \cdots + c_mc_m$ for $v_i \in E(\lambda_i, T)$. Then:

$$\begin{aligned} w &= Tv \\ &= T(c_1v_1 + c_2v_2 + \cdots + c_mv_m) \\ &= T(c_2v_2 + \cdots + c_mv_m) && \text{(Why?)} \\ &= c_2\lambda_2v_2 + \cdots + c_m\lambda_mv_m, \end{aligned}$$

so it follows that $w \in E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(⊇) Let $v_2 + \cdots + v_m \in E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T)$. Then

$$T\left(\frac{1}{\lambda_2}v_2 + \cdots + \frac{1}{\lambda_m}v_m\right) = v_2 + \cdots + v_m,$$

since $\lambda_i \neq 0$ for all $i = 2, \dots, m$. Hence $v_2 + \cdots + v_m \in \operatorname{range} T$. ♢

Exercise (6.A.7)**.** Suppose $u, v \in V$. Prove that $\|au + bv\| = \|ub + av\|$ for all $a, b \in \mathbb{R}$ if and only if $\|u\| = \|v\|$.

Solution. The forward direction is straightforward. (Why? Pick convenient values of a and b.)

Now suppose that $\|u\| = \|v\|$ and let $a, b \in \mathbb{R}$ be arbitrary. It is sufficient to show that $\|au + bv\|^2 = \|bu + av\|^2$. (Why?) You should fill in the missing steps in the following computation:

$$\begin{aligned}\|au + bv\|^2 &= \langle au + bv, au + bv\rangle \\ &= a^2\|u\|^2 + b^2\|v\|^2 + \langle av, bu\rangle + \langle bu, av\rangle \\ &= a^2\|v\|^2 + b^2\|u\|^2 + \langle av, bu\rangle + \langle bu, av\rangle \qquad \text{(Why?)} \\ &= \langle bu + av, bu + av\rangle.\end{aligned}$$

◊

Exercise (6.B.1)**.**

(a) Suppose $\theta \in \mathbb{R}$. Show that the two lists $(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$ and $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$ are orthonormal bases of $\mathbb{R}^2$.

(b) Show that any orthonormal basis of $\mathbb{R}^2$ is of the form given by one of the two lists of part (a).

Solution.

(a) Note that we are equipping $\mathbb{R}^2$ with the usual Euclidean inner product (i.e., the dot product). For each list, label the elements α, β. It is a straightforward computation to verify that:

- $\langle \alpha, \alpha\rangle = 1$,
- $\langle \beta, \beta\rangle = 1$,
- $\langle \alpha, \beta\rangle = 0$,

as you should verify. Why is this sufficient to show that each list α, β is an orthonormal basis?

(b) What does an orthonormal basis of $\mathbb{R}^2$ look like? Argue geometrically. (Note that the two vectors must lie on the unit circle. What else can you say?)

◊

Exercise (6.C.11)**.** In $\mathbb{R}^4$ with the Euclidean inner product, let

$$U = \operatorname{span}\big((1, 1, 0, 0), (1, 1, 1, 2)\big).$$

Find $u \in U$ such that $\|u - (1, 2, 3, 4)\|$ is as small as possible.

Solution. By the result of 6.55(i), we are required to compute

$$P_U v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2,$$

where $v - (1, 2, 3, 4)$ and e_1, e_2 is an orthonormal basis of U. You should compute that the Gram-Schmidt yields an orthonormal basis

$$e_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right), \ e_2 = \left(0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right).$$

Then you can compute that

$$P_U v = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5} \right).$$

◊

Exercise (7.A.2)**.** Suppose $T \in \mathcal{L}(V)$. Prove that $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

Solution. It is equivalent to show that: $\lambda \in \mathbb{F}$ is not an eigenvalue of T if and only if $\overline{\lambda}$ is not an eigenvalue of T^*. Then:

$$\begin{aligned}
& \lambda \text{ is not an eigenvalue of } T \\
\iff & T - \lambda I \text{ is invertible} && \text{(Why?)} \\
\iff & S(T - \lambda I) = I \text{ for some } S \in \mathcal{L}(V) && \text{(Why? Note } S \text{ is an inverse)} \\
\iff & (T^* - \overline{\lambda} I) S^* = I && \text{(Why?)} \\
\iff & T^* - \overline{I} \text{ is invertible} \\
\iff & \overline{\lambda} \text{ is not an eigenvalue of } T^*
\end{aligned}$$

◊