Math 3GR3, Tutorial 10

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Topics: First isomorphism theorem, applications.

Question 1 (Modified from Judson exercise 11.4.13)**.** Consider the following statement:

$\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}$ are isomorphic as groups.

(a) ChatGPT proved this statement as follows. Find the mistake(s).

Proof. To show that $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}$ as groups, we need to find a group isomorphism between them. Let's define a map $\phi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}$ as follows:

$$\phi\left(\frac{a}{b} + \mathbb{Z}\right) = \frac{a}{b},$$

where $\frac{a}{b} \in \mathbb{Q}$ is a rational number, and we identify it with its coset $\frac{a}{b} + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$.

First, we need to show that ϕ is well-defined. Suppose $\frac{a}{b} + \mathbb{Z} = \frac{c}{d} + \mathbb{Z}$. This means $ad = bc$. Since b and d are coprime (having no common factors other than 1), it follows that b divides c, say $c = bk$. Now, we have $\frac{a}{b} = \frac{bk}{d}$, and since b and d are coprime, b divides k, say $k = bm$. Thus, $\frac{a}{b} = \frac{b \cdot bm}{d} = \frac{b^2 m}{d}$, which shows that $\frac{a}{b} = \frac{b^2 m}{d} + \mathbb{Z}$. So, the choice of representative does not matter, and ϕ is well-defined.

Next, we need to show that ϕ is a group homomorphism. Let $\frac{a}{b} + \mathbb{Z}$ and $\frac{c}{d} + \mathbb{Z}$ be two elements in $\mathbb{Q}/\mathbb{Z}$. Then,

$$\begin{aligned}
\phi\left(\left(\frac{a}{b} + \mathbb{Z}\right) + \left(\frac{c}{d} + \mathbb{Z}\right)\right) &= \phi\left(\frac{a}{b} + \frac{c}{d} + \mathbb{Z}\right) \\
&= \phi\left(\frac{ad + bc}{bd} + \mathbb{Z}\right) \\
&= \frac{ad + bc}{bd} \\
&= \frac{a}{b} + \frac{c}{d} \\
&= \phi\left(\frac{a}{b} + \mathbb{Z}\right) + \phi\left(\frac{c}{d} + \mathbb{Z}\right).
\end{aligned}$$

This shows that ϕ preserves the group operation.

Now, let's show that ϕ is injective. Suppose $\phi\left(\frac{a}{b}+\mathbb{Z}\right) = \phi\left(\frac{c}{d}+\mathbb{Z}\right)$. This implies $\frac{a}{b} = \frac{c}{d}$, which, in turn, means $ad = bc$. Since b and d are coprime, this forces $a = c$ and $b = d$, so $\frac{a}{b}+\mathbb{Z} = \frac{c}{d}+\mathbb{Z}$. Thus, ϕ is injective.

Finally, let's show that ϕ is surjective. For any $\frac{a}{b} \in \mathbb{Q}$, we have $\phi\left(\frac{a}{b}+\mathbb{Z}\right) = \frac{a}{b}$, so ϕ is surjective.

Since ϕ is a well-defined, injective, and surjective group homomorphism, it is an isomorphism, and we can conclude that $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}$ as groups. □

(b) Determine whether the statement is true or false. If it is true, fix ChatGPT's proof to show that they are isomorphic. If it is false, give a proof that they are not isomorphic.

Question 2. Recall that $\mathbb{R}^*$ is a multiplicative group and $\mathbb{R}$ is an additive group. Show that $\mathbb{R}^*/\{\pm 1\} \cong \mathbb{R}$.

Question 3 (Judson 16.6.34)**.** Let p be a prime integer. Prove that the **ring of integers localized at** p, given by

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \,\middle|\, \gcd(b,p) = 1 \right\},$$

is a ring, and moreover, that it is an integral domain. Determine the characteristic of $\mathbb{Z}_{(p)}$.