Math 3GR3, Tutorial 11

Mike Cummings

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Topics: Rings. Integral domains, etc. Isomorphisms.

Question 1. Give an example of. . .

(a) a noncommutative ring;
(b) a ring without (multiplicative) identity (AKA a rng);
(c) a ring with identity that is not a division ring;
(d) a commutative ring with identity that is not an integral domain;
(e) an integral domain that is not a field.

Question 2. Show that $\mathbb{R}[x]/\langle x^2+1\rangle \cong \mathbb{C}$. *[Hint: recall from linear algebra that if z is a root of a polynomial in $\mathbb{R}[x]$, then so is $\overline{z}$.]*

Question 3 (Judson 16.6.26)**.** Let R be an integral domain. If the only ideals of R are $\{0\}$ and R itself, then show that R is a field.

Question 4. A **principal ideal domain (PID)** is an integral domain D for which every ideal $I \subseteq D$ can be generated by a single element, e.g., there exists some $a \in D$ such that $I = \langle a \rangle$. Show that the integers $\mathbb{Z}$ form a PID.

Think about how you might adapt your argument to show that $\mathbb{R}[x]$ is a PID.

Question 5 (Judson 16.6.27)**.** Let R be a commutative ring. An element a of R is called **nilpotent** if $a^n = 0$ for some positive integer n. Show that the set of all nilpotent elements is an ideal of R.