Math 3GR3, Tutorial 1

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September 12, 2023

Topics: Equivalence relations and partitions. Euclidean algorithm and greatest common divisor.

Question 1. Recall that an equivalence relation is a relation on a nonempty set that is reflexive, symmetric, and transitive. Explain why the following relations are not equivalence relations.

(a) $x \sim y$ in $\mathbb{R}$ if $x \neq y$

(b) $x \sim y$ in $\mathbb{C}$ if $x \leq y$

(c) Let $\mathrm{Mat}_n(\mathbb{Q})$ denote $n \times n$ matrices with entries in $\mathbb{Q}$. Define $A \sim B$ in $\mathrm{Mat}_n(\mathbb{Q})$ if $\det(AB) < 0$.

Question 2 (Judson Chapter 1, Exercise 29. The projective real line)**.** Define a relation on $\mathbb{R}^2 \setminus \{(0,0)\}$ by letting $(x_1, y_1) \sim (x_2, y_2)$ if there exists a nonzero real number λ such that $(x_1, y_1) = (\lambda x_2, \lambda y_2)$.

(a) Prove that $\sim$ defines an equivalence relation on $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) What are the corresponding equivalence classes?

Question 3 (Lakins Exercise 6.2.1(c))**.** Compute $\gcd(7776, 16650)$ and find integers x, y such that $7776x + 16650y = \gcd(a, b)$.

Question 4 (Judson 2.4.16)**.** Let a and b be nonzero integers. If there exist integers r and s such that $ar + bs = 1$, show that a and b are relatively prime.

Question 5 (Judson 2.4.24)**.** If $d = \gcd(a, b)$ and $m = \mathrm{lcm}(a, b)$, prove that $dm = |ab|$.

Question 6. Let p be a prime number.

(a) (Lakins Exercise 6.3.6) If i is an integer satisfying $0 < i < p$, show that $\binom{p}{i} \equiv 0 \mod p$. That is, show that p divides $\binom{p}{i}$.

(b) Give an example to show that (a) fails if p is not prime.

(c) (Freshman's dream) Let a and b be integers. Using (a), show that $(a+b)^p \equiv a^p + b^p \mod p$. [Hint: binomial theorem.]