Math 2LA3 Assignment 3

1. Let $\vec{u} = (1, 0, -3)$ and $\vec{v} = (0, 1, 2)$. Use the definition of the dot product to find a vector $\vec{w}$ in $\mathbb{R}^3$ such that $\vec{u} \cdot \vec{w} = 0$, $\vec{v} \cdot \vec{w} = 0$, and such that the distance between $\vec{v}$ and $\vec{w}$ is $\sqrt{47}$.

2. Let N be the last digit of your student number. Consider the following vectors:

$$\vec{w}_1 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 5 \\ 16 \\ N \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 29 \\ 46 \\ 33 \end{bmatrix}.$$

   (a) Is $\vec{v}$ in the span of $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$? If so, find scalars a, b, c such that $\vec{v} = a\vec{w}_1 + b\vec{w}_2 + c\vec{w}_3$. If not, explain why no such scalars exist.

   (b) Let W be the subspace of $\mathbb{R}^3$ defined by $W = \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3)$. Does the list $\vec{w}_1, \vec{w}_2, \vec{w}_3$ form a basis for W? Explain.

3. Let N be the last digit of your student number and consider the following matrix A. Use Gram-Schmidt to find an orthonormal basis for col(A), the column span of A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & N \end{bmatrix}.$$

4. What happens if you apply the Gram-Schmidt algorithm with a linearly *dependent* list of vectors? Give a geometric explanation as to why this happens.

   Hint: Try an example and see what happens. You do not need to include your example in your answer.

5. Given the following matrix A and vector $\vec{b}$, which vector $\vec{y}$ in col(A) best approximates $\vec{b}$? Do not use QR factorization or the normal equations.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

6. Use QR factorization to solve the least squares problem defined by A and $\vec{b}$. That is, find the vector $\vec{x}$ that minimizes the distance $\|A\vec{x} - \vec{b}\|$.

$$A = \begin{bmatrix} 0 & -1 \\ 4 & 9 \\ 0 & 2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

7. Select **all** of following pairs Q and R that form a QR factorization for some matrix A.

   (i) $$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

(ii) $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & \sqrt{2}/\sqrt{3} \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 0 & 7 \\ 0 & 0 & 11 \end{bmatrix}$

(iii) $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ -1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & \sqrt{3}/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$

(iv) $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -3/5 \\ 0 & -1/\sqrt{2} & 4/5 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{11} & 2\sqrt{11} & 3\sqrt{11} \\ 0 & 525 & 600 \\ 0 & 0 & \pi \end{bmatrix}$

(v) $Q = \begin{bmatrix} 1/\sqrt{14} & -7/3\sqrt{10} \\ -2/\sqrt{14} & 4/3\sqrt{10} \\ 3/\sqrt{14} & 5/3\sqrt{10} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$