Math 2LA3 Assignment 4

1. Find the vector $\vec{x}$ for which $A\vec{x}$ best approximates $\vec{b}$ using the normal equations and find the least squares error $e = \|A\vec{x} - \vec{b}\|$.

   Round the least squares error to 2 decimal places, but keep all other numbers exact.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

2. This question is a generalization of the lines of best fit $\beta_0 + \beta_1 x$ we have seen in lecture.

   Suppose that a dataset has two input variables x_i, y_i and one output variable z_i. We wish to model the output z_i with a plane of best fit $z = \beta_0 + \beta_1 x + \beta_2 y$.

   Find the coefficients $\beta_0, \beta_1, \beta_2$ that give the plane of best fit for the following dataset.

x_i	0	0	0	1	1	1
y_i	1	2	3	1	2	3
z_i	5	3	7	4	8	7

3. Consider the diagonal matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Consider the corresponding quadratic form $Q(\vec{x}) = \vec{x}^T D\vec{x}$.

   Version 1.

   (a) Argue that $Q(\vec{x}) \leq \lambda_1$ for any vector $\vec{x}$ with $\|\vec{x}\| = 1$.

   (b) Conclude that $\max\limits_{\|\vec{x}\|=1} Q(\vec{x}) = \lambda_1$. That is, find a vector $\vec{x}$ with $\|\vec{x}\| = 1$ for which $Q(\vec{x}) = \lambda_1$.

   Version 2.

   (a) Argue that $Q(\vec{x}) \geq \lambda_n$ for any vector $\vec{x}$ with $\|\vec{x}\| = 1$.

   (b) Conclude that $\min\limits_{\|\vec{x}\|=1} Q(\vec{x}) = \lambda_n$. That is, find a vector $\vec{x}$ with $\|\vec{x}\| = 1$ for which $Q(\vec{x}) = \lambda_n$.

4. Orthogonally diagonalize the following matrix.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

5. (a) Argue that if A is orthogonally diagonalizable, then A must be symmetric.
   Hint: Compute A^T.

   (b) Give an example of a diagonalizable matrix that is not orthogonally diagonalizable.

6. Find a symmetric matrix A whose associated quadratic form is

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 6x^2 + 7xy + 8y^2$$

   or explain why no such A exists.

7. Consider the quadratic form $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$. Find an orthogonal matrix P whose change of coordinates $\vec{x} = P\vec{y}$ transforms the quadratic form $Q(\vec{x})$ into one with no cross-terms.

8. **True/false.** Select all of the following statements that are **true**.

   (i) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is orthogonal and c_1, c_2, c_3 are real scalars, then $c_1\vec{v}_1, c_2\vec{v}_2, c_3\vec{v}_3$ is orthogonal.

   (ii) If $\vec{u}$ and $\vec{v}$ are orthogonal and nonzero, then they are linearly independent.

   (iii) Let A be a symmetric matrix. If $\vec{u}$ and $\vec{v}$ are nonzero vectors such that $A\vec{u} = \vec{u}$ and $A\vec{v} = \pi\vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.

   (iv) Let A be a symmetric matrix. If every entry of A is strictly positive, then the quadratic form associated to A is always nonnegative. That is, $\vec{x}^T A\vec{x} \geq 0$ for all $\vec{x}$.