Math 2LA3 Assignment 5

1. Find the maximum and minimum of the quadratic form $Q(\vec{x}) = 2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$ subject to the constraint $\|\vec{x}\|^2 = 1$. At which points $\vec{x}$ do the maxima/minima occur?

2. Classify the following quadratic forms. Justify your answers.

   (a) $9x_1^2 - 8x_1x_2 + 3x_2^2$

   (b) $3y_1^2 + 2y_1y_2 + 2y_1y_3 + 4y_2y_3$

3. Give an example of a quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ that is...

   Version 1.

   (a) positive definite

   (b) negative semidefinite but not negative definite

   (c) indefinite.

   No justification needed.

   Version 2.

   (a) negative definite

   (b) positive semidefinite but not positive definite

   (c) indefinite.

   No justification needed.

4. Sketch the graph of a quadratic form that is...

   Version 1.

   (a) positive definite

   (b) positive semidefinite but not positive definite

   (c) indefinite.

   No justification needed.

   Version 2.

   (a) negative definite

   (b) negative semidefinite but not negative definite

   (c) indefinite.

5. Using methods from Math 2LA3, compute the rank of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -9 & 6 & -3 \\ 4 & 5 & 6 \end{bmatrix}$$

   You may use an online calculator to compute the eigenvalues of $A^T A$.

6. Compute a singular value decomposition of A.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

7. Given the following singular value decomposition $U\Sigma V^T$ of a matrix A, find the least squares solution for the system $A\vec{x} \approx \vec{b}$.

$$U = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 8 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

8. **True/false.** Select all of the following statements that are true.
   (a) The expression $\|\vec{x}\|^2$ is a quadratic form.
   (b) If A is a skew-symmetric matrix, then the polynomial $\vec{x}^T A\vec{x}$ has no cross terms. (Skew-symmetric matrices satisfy $A^T = -A$.)
   (c) Singular value decompositions are unique.
   (d) Every matrix has a pseudoinverse.