Webs & Smooth Components
of
Two Column Springer Fibers
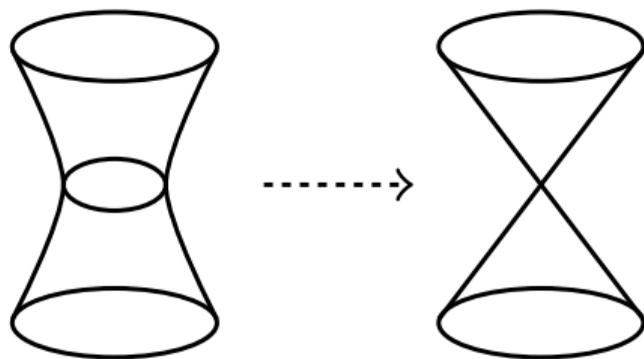
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Springer ('69) gave a resolution of singularities [...] of the nilpotent cone in $\mathfrak{sl}_n$

Spaltenstein ('76). The fiber of the resolution at η

$$\text{"Springer fiber"} \quad S_\eta = \left\{ \text{flags } V_\bullet \mid \eta \cdot V_i \subseteq V_{i-1} \text{ for all } i \right\}$$

is ■ equidimensional, ■ connected, ■ components indexed by $\text{SYT}(\eta)$
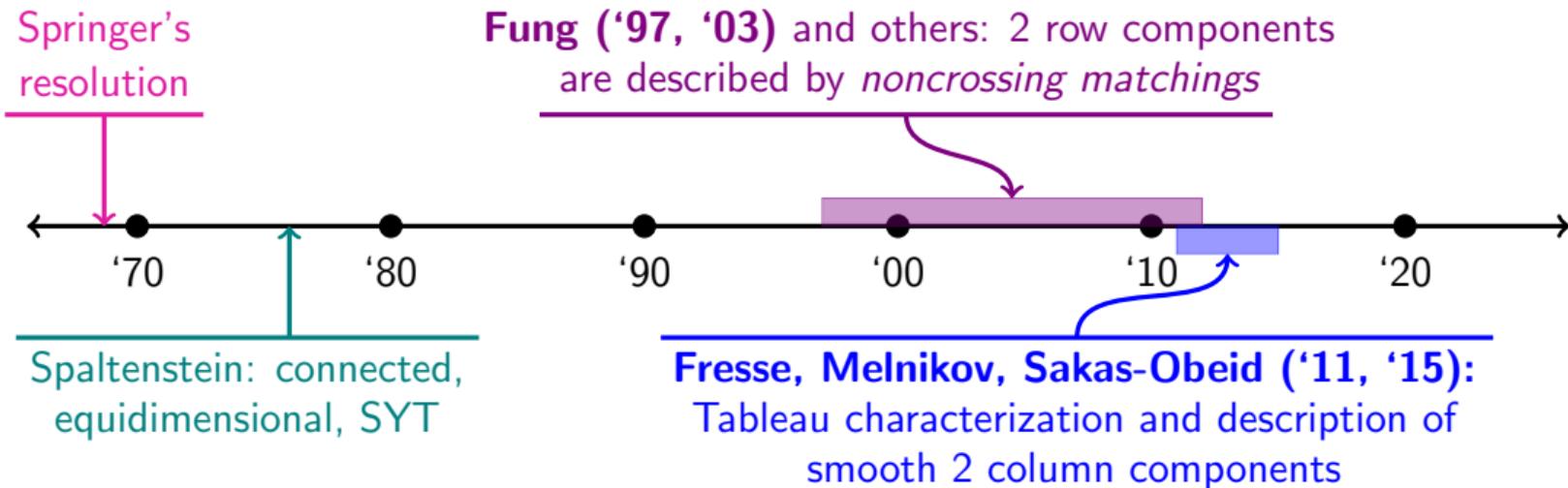
Combinatorics of Springer Fibers S_η

- Components are indexed by standard Young tableaux of shape η

- Equidimensional of dimension $\sum_i (i-1)\eta_i$

0	0	0
1	1	1
2	2	2

$\leadsto \quad S_{(3,3,3)}$ is 9-dimensional

- $H^\star(S_\eta)$ carries an $\mathfrak{S}_n$-action and
    - gives rise to Hall–Littlewood polynomials
    - top graded piece of $H^\star(S_\eta)$ is the Specht module S^η

Geometry of Springer Fibers

Springer's resolution

Fung ('97, '03) and others: 2 row components
are described by *noncrossing matchings*

'70   '80   '90   '00   '10   '20

Spaltenstein: connected,
equidimensional, SYT

Fresse, Melnikov, Sakas-Obeid ('11, '15):
Tableau characterization and description of
smooth 2 column components

2 Column Tableau Descriptions

Let $T \in \mathrm{SYT}(k \times 2)$ with corresponding Springer component F_T. Define:

$$\tau(T) = \{j \in \mathrm{col}_1(T) \mid j+1 \in \mathrm{col}_2(T)\}$$

Then,

$$F_T \text{ smooth} \quad \overset{\text{Fresse, Melnikov ('11)}}{\Longleftrightarrow} \quad \begin{array}{l} |\tau(T)| < 3 \text{ or} \\ |\tau(T)| = 3 + \text{another condition} \end{array}$$

Fresse, Melnikov, Sakas-Obeid ('15). If F_T is smooth, it is an *iterated fibre bundle* with base

$$\big(\mathrm{Fl}(a+b) \times \mathrm{Fl}(b+c), \ \mathrm{Gr}(a, a+c), \ \mathbf{P}^b, \ \ldots, \ \mathbf{P}^{k-1} \big),$$

where:

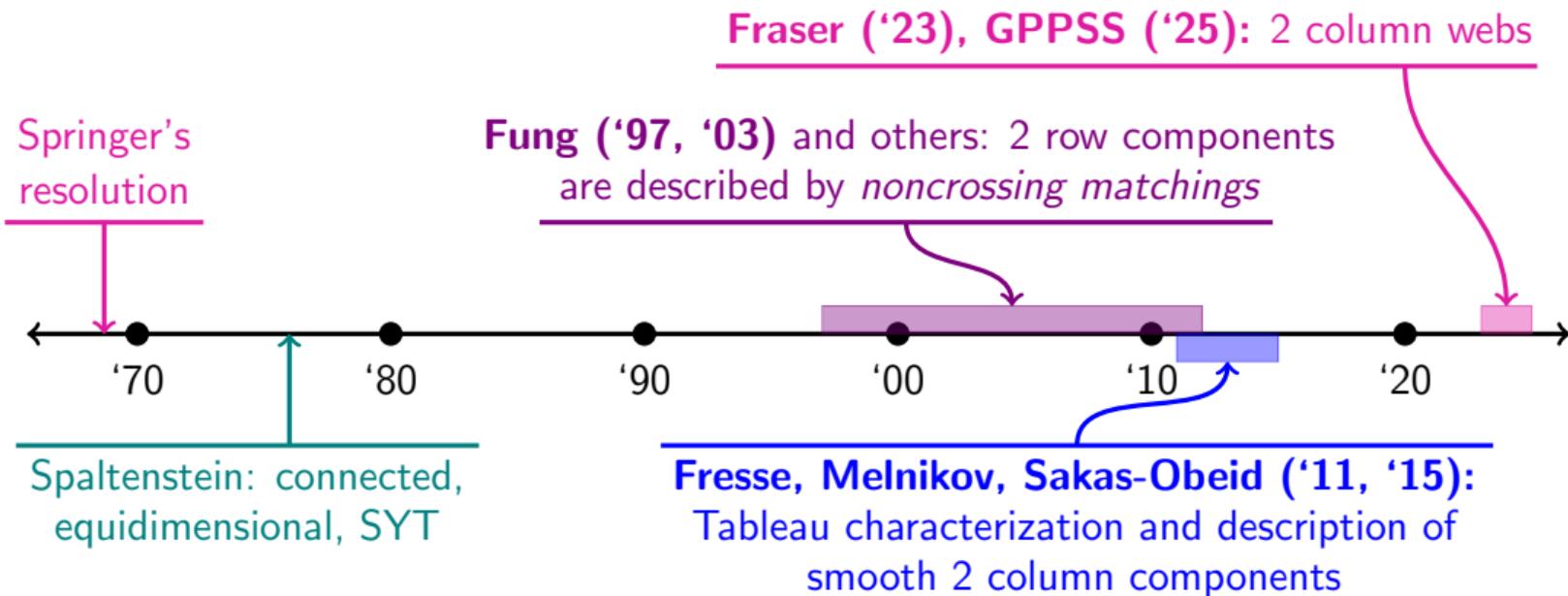| τ	a	b	c
$\{\alpha\}$	$\alpha - k$	k	$k - \alpha$
$\{\alpha < \beta\}$	$k - (\beta - \alpha)$	$\beta - k$	$k - \alpha$
$\{\alpha < \beta < \gamma\}$	$k - (\gamma - \beta)$	$(\gamma - \alpha) - k$	$k - (\beta - \alpha)$

2 Column Tableau Examples

1	5
2	7
3	8
4	10
6	11
9	12

singular

1	6
2	7
3	8
4	10
5	11
9	12

smooth

$\left(\text{Fl}(5) \times \text{Fl}(4), \ \text{Gr}(2,3), \ \mathbf{P}^3, \mathbf{P}^4, \mathbf{P}^5 \right)$

Geometry of Springer Fibers

Fraser ('23), GPPSS ('25): 2 column webs

Fung ('97, '03) and others: 2 row components are described by *noncrossing matchings*

Springer's resolution

'70   '80   '90   '00   '10   '20

Spaltenstein: connected, equidimensional, SYT

Fresse, Melnikov, Sakas-Obeid ('11, '15): Tableau characterization and description of smooth 2 column components

2 Column Web Examples
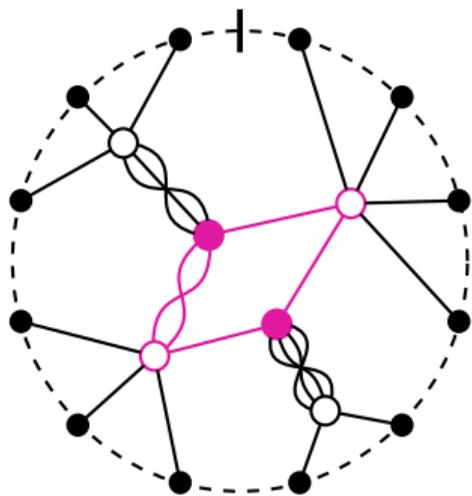
cycle $\implies$ singular
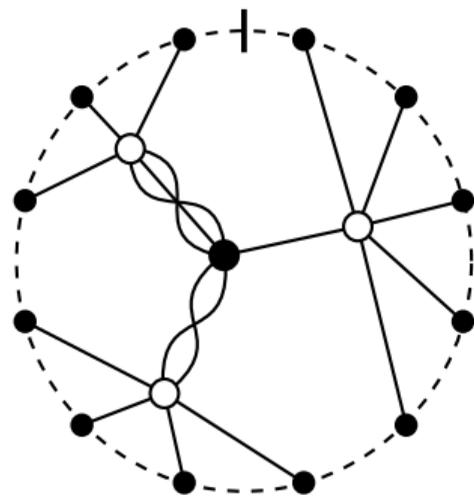
acyclic $\implies$ smooth

$\big(\, \mathrm{Fl}(5) \times \mathrm{Fl}(4), \;\; \mathrm{Gr}(2,3), \;\; \mathbf{P}^3, \mathbf{P}^4, \mathbf{P}^5 \big)$

Summary

[C. 26+] Smooth components of two column Springer fibers...

- correspond to acyclic webs
- geometry described by claw sizes
- Poincaré polynomial is rotation/reflection invariant
- $k + 2\binom{k}{3}$ smooth components for $S_{(2^k)}$

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