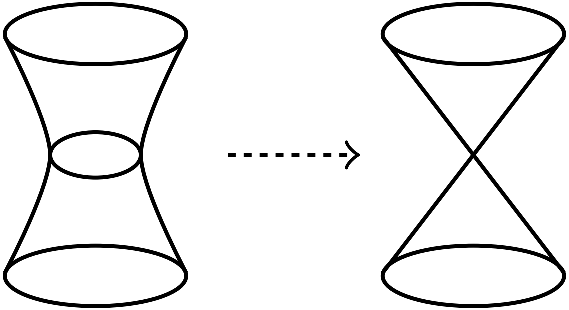
Webs & Smooth Components of Two Column Springer Fibers

Mike Cummings
University of Waterloo

CMND 2026 Thematic Program:
Algebraic Combinatorics and Applications
7 July 2026

Springer's Resolution

Springer ('69) Gives a resolution of singularities of the nilpotent cone in $\mathfrak{sl}_n$



fibers $S_\eta = \{\text{flags } V_\bullet \mid \eta \cdot V_i \subseteq V_{i-1} \text{ for all } i\}$

Springer Fibers

The fiber of the resolution at η

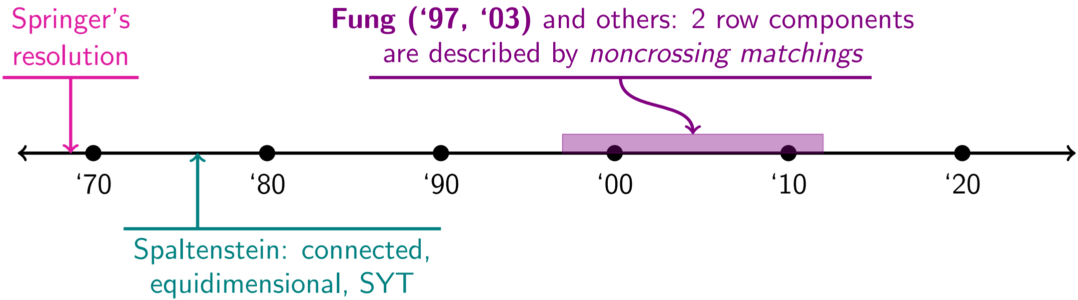
$$\text{“Springer fiber''} \quad S_\eta = \{\text{flags } V_\bullet \mid \eta \cdot V_i \subseteq V_{i-1} \text{ for all } i\}$$

Spaltenstein ('76) S_η is equidimensional; connected; components indexed by SYT(η)

Springer ('76) $H^\star(S_\eta)$ carries an $\mathfrak{S}_n$-action; $H^{\text{top}}(S_\eta)$ is the Specht module S^η

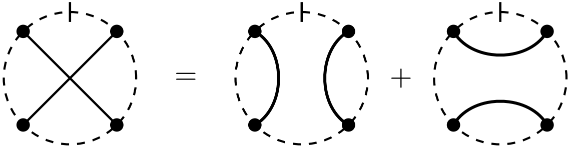
Hotta, Springer ('76) $H^\star(S_\eta)$ gives rise to Hall–Littlewood polynomials

Geometry of Springer Fibers



Web Bases

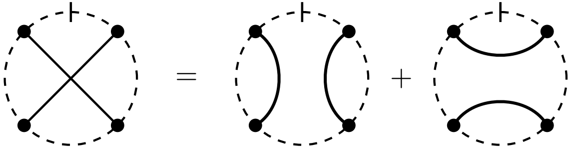
Noncrossing matchings are an $\mathfrak{sl}_2$ **web basis** for $\mathrm{Inv}_{\mathfrak{sl}_2}(V^{\otimes n})$



Kuperberg ('96) $\mathfrak{sl}_3$ web basis for vector spaces of tensor invariants of representations

Web Bases

Noncrossing matchings are an $\mathfrak{sl}_2$ **web basis** for $\mathrm{Inv}_{\mathfrak{sl}_2}(V^{\otimes n})$



Kuperberg ('96) $\mathfrak{sl}_3$ web basis for vector spaces of tensor invariants of representations

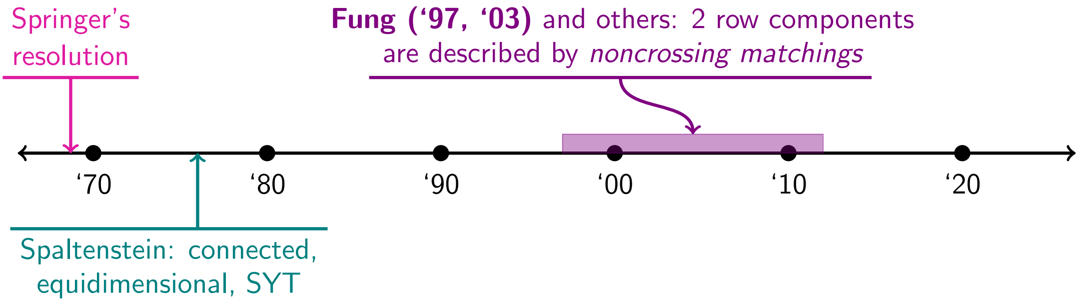
Many applications:

- Khovanov ('00) categorification of the Jones polynomial
- Petersen, Pylyavskyy, Rhoades ('08) exhibits cyclic sieving
- Fomin, Pylyavskyy ('12) cluster algebras

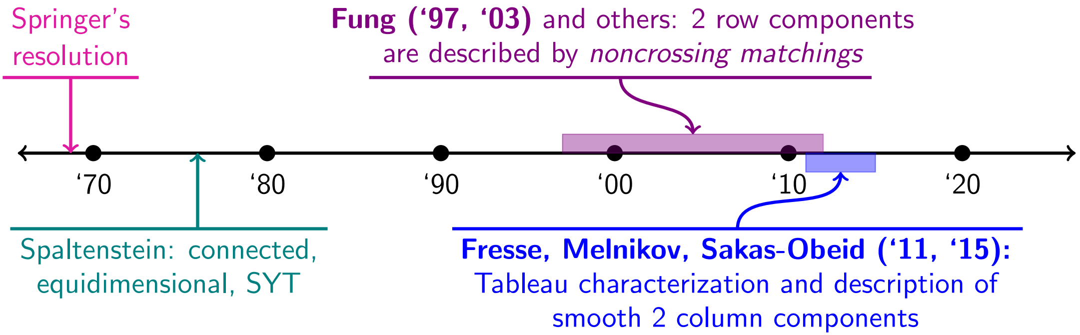
and more!

Open: describing a "good" $\mathfrak{sl}_r$ web basis for $r \geq 5$

Geometry of Springer Fibers



Geometry of Springer Fibers



2 Column Tableau Descriptions

Let $T \in \mathrm{SYT}(k \times 2)$ with corresponding Springer component F_T. Define:

$$\mathrm{Asc}(T) = \{j \in \mathrm{col}_1(T) \mid j+1 \in \mathrm{col}_2(T)\}$$

Then,

$$F_T \text{ smooth} \quad \overset{\textbf{Fresse, Melnikov ('11)}}{\Longleftrightarrow} \quad \begin{array}{c} \# \mathrm{Asc}(T) < 3 \text{ or} \\ \# \mathrm{Asc}(T) = 3 + \text{another condition} \end{array}$$

2 Column Tableau Descriptions

Let $T \in \mathrm{SYT}(k \times 2)$ with corresponding Springer component F_T. Define:

$$\mathrm{Asc}(T) = \left\{ j \in \mathrm{col}_1(T) \mid j+1 \in \mathrm{col}_2(T) \right\}$$

Then,

$$F_T \text{ smooth} \quad \overset{\textbf{Fresse, Melnikov ('11)}}{\Longleftrightarrow} \quad \begin{array}{c} \#\,\mathrm{Asc}(T) < 3 \text{ or} \\ \#\,\mathrm{Asc}(T) = 3 + \text{another condition} \end{array}$$

Fresse, Melnikov, Sakas-Obeid ('15). If F_T is smooth, it is an *iterated fibre bundle* with base

$$\big(\, \mathrm{Fl}(a+b) \times \mathrm{Fl}(b+c),\ \ \mathrm{Gr}(a, a+c),\ \mathbf{P}^b,\ \ \ldots,\ \mathbf{P}^{k-1} \big),$$

2 Column Tableau Descriptions

Let $T \in \mathrm{SYT}(k \times 2)$ with corresponding Springer component F_T. Define:

$$\mathrm{Asc}(T) = \{j \in \mathrm{col}_1(T) \mid j+1 \in \mathrm{col}_2(T)\}$$

Then,

$$F_T \text{ smooth} \quad \overset{\text{Fresse, Melnikov ('11)}}{\Longleftrightarrow} \quad \begin{array}{c} \#\,\mathrm{Asc}(T) < 3 \text{ or} \\ \#\,\mathrm{Asc}(T) = 3 + \text{another condition} \end{array}$$

Fresse, Melnikov, Sakas-Obeid ('15). If F_T is smooth, it is an *iterated fibre bundle* with base

$$\big(\,\mathrm{Fl}(a+b) \times \mathrm{Fl}(b+c),\ \ \mathrm{Gr}(a, a+c),\ \mathbf{P}^b,\ \ \ldots,\ \mathbf{P}^{k-1}\big),$$

where:

Asc	a	b	c
$\{\alpha\}$	$\alpha - k$	k	$k - \alpha$
$\{\alpha < \beta\}$	$k - (\beta - \alpha)$	$\beta - k$	$k - \alpha$
$\{\alpha < \beta < \gamma\}$	$k - (\gamma - \beta)$	$(\gamma - \alpha) - k$	$k - (\beta - \alpha)$

2 Column Tableau Examples

1	5
2	7
3	8
4	10
6	11
9	12

1	6
2	7
3	8
4	10
5	11
9	12

2 Column Tableau Examples

1	5
2	7
3	8
4	10
6	11
9	12

singular

1	6
2	7
3	8
4	10
5	11
9	12

smooth

2 Column Tableau Examples

1	5
2	7
3	8
4	10
6	11
9	12

singular

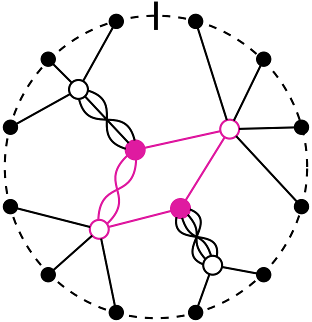
1	6
2	7
3	8
4	10
5	11
9	12

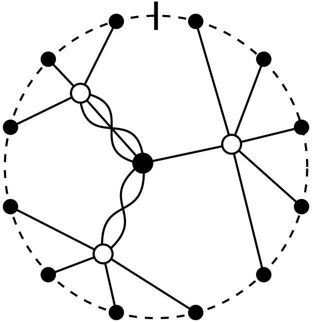
smooth

$$\big(\,\mathrm{Fl}(5)\times\mathrm{Fl}(4),\;\;\mathrm{Gr}(2,3),\;\;\mathbf{P}^3,\mathbf{P}^4,\mathbf{P}^5\big)$$

2 Column Web Examples

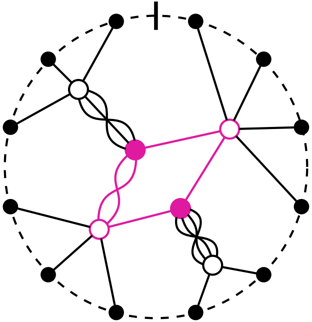
Using the two column web basis of Fraser ('23) and Gaetz, Pechenik, Pfannerer, Striker, and Swanson ('25)



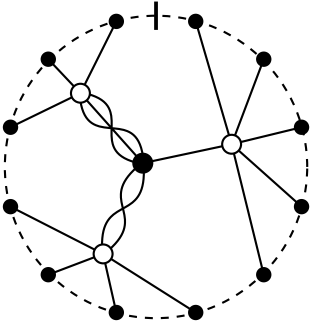


2 Column Web Examples

Using the two column web basis of Fraser ('23) and Gaetz, Pechenik, Pfannerer, Striker, and Swanson ('25)



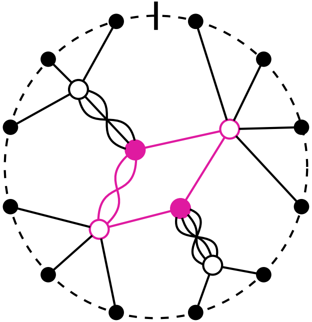
cycle $\Longrightarrow$ singular



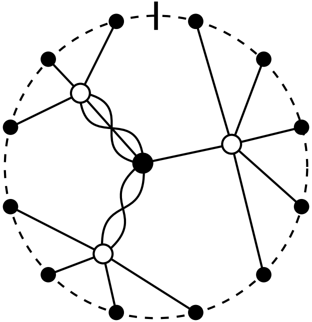
acyclic $\Longrightarrow$ smooth

2 Column Web Examples

Using the two column web basis of Fraser ('23) and Gaetz, Pechenik, Pfannerer, Striker, and Swanson ('25)



cycle $\Longrightarrow$ singular



acyclic $\Longrightarrow$ smooth

$\big(\mathrm{Fl}(5) \times \mathrm{Fl}(4),\ \ \mathrm{Gr}(2,3),\ \ \mathbf{P}^3, \mathbf{P}^4, \mathbf{P}^5\big)$

Summary

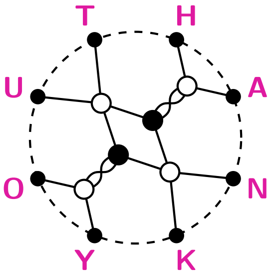
Theorem (C. 26+). Smooth components of two column Springer fibers...

- correspond to acyclic webs
- geometry described by claw sizes
- Poincaré polynomial is rotation/reflection invariant
- $k + 2\binom{k}{3}$ smooth components for $S_{(2^k)}$

Summary

Theorem (C. 26+). Smooth components of two column Springer fibers...

- correspond to acyclic webs
- geometry described by claw sizes
- Poincaré polynomial is rotation/reflection invariant
- $k + 2\binom{k}{3}$ smooth components for $S_{(2^k)}$



arXiv 2602.16910